18.03 MIDTERM 1 - SOLUTIONS

October 2, 2019 (50 minutes)

PROBLEM 1

(1) Use Gaussian elimination to write the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ -1 & 2 & -2 & -1 & 2 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}$ as $A = LU$, where L is a lower triangular 4×4 matrix and U is a 4×5 matrix in row echelon form. *(15 pts)*

Solution: Using Gaussian elimination we get:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can rewrite these steps as multiplications by various elimination matrices and diagonal matrices. The first step is given by $E_{21}^{(1)}$, the second by $E_{32}^{(1)}$ and the third by $E_{43}^{(-2)}$. Thus we get:

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad L = E_{21}^{(-1)} E_{32}^{(-1)} E_{43}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(2) Express the matrix L in part (1) as a product of elimination matrices $E_{ij}^{(\lambda)}$ for various $i > j$ and constants λ. Then do the same for its inverse L^{-1}. *(10 pts)*

Solution: Note from the computation above that we have written

$$L = E_{21}^{(-1)} E_{32}^{(-1)} E_{43}^{(2)}$$

Then using this and the fact that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, we can compute

$$L^{-1} = E_{43}^{(-2)} E_{32}^{(1)} E_{21}^{(1)}$$

(3) With the same notations as in part (1), find a permutation matrix P such that $PA' = LU$, where $A' = \begin{bmatrix} -1 & 2 & -2 & -1 & 2 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & -2 & 0 & 0 & -1 \end{bmatrix}$. *(5 pts)*

Solution: Note that A' is just A with the first two rows swapped and the last two rows swapped, so the permutation matrix we need is

$$P = P_{12}P_{34} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

PROBLEM 2

Consider the system of equations:

$$\begin{cases} a - 3b + 6c - d = 1 \\ -2a + 5b - 11c + 2d = -2 \end{cases} \qquad (*)$$

(1) Write the system as $A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{b}$ for a suitably chosen 2×4 matrix A and 2×1 vector $\mathbf{b}$. *(5 pts)*

Solution:We can rewrite the system of equations as:

$$\begin{bmatrix} 1 & -3 & 6 & -1 \\ -2 & 5 & -11 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(2) Use Gauss-Jordan elimination to put A in <u>reduced row echelon form</u>. *(10 pts)*

Solution: First add 2 times row 1 to row 2:

$$\begin{bmatrix} 1 & -3 & 6 & -1 \\ -2 & 5 & -11 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & 6 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Then multiply row 2 by -1, to get all pivots equal to 1:

$$\begin{bmatrix} 1 & -3 & 6 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & 6 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Finally, add 3 times row 2 to row 1:

$$\begin{bmatrix} 1 & -3 & 6 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

(3) What are basis vectors for the nullspace of A? What is its dimension? *(10 pts)*

Solution: The nullspace is the set of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that:

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \qquad \Leftrightarrow \qquad \begin{cases} a = -3c + d \\ b = c \end{cases}$$

The pivot columns are 1 and 2, and the free columns are 3 and 4. Recall that basis vectors are given by setting (c, d) equal to either $(1, 0)$ or $(0, 1)$, and using the equations above to solve for a and b:

$$\text{a collection of basis vectors of } N(A) \text{ are } \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the dimension of $N(A)$ is 2.

(4) What is the general solution of the system $(*)$? *(10 pts)*

Solution: A particular solution can be obtained by setting the free variables c and d equal to 0, and solving for the pivot variables:

$$\begin{cases} a - 3b = 1 \\ -2a + 5b = -2 \end{cases}$$

You can solve this 2×2 system in a number of ways, or just notice that $a = 1$, $b = 0$ is a solution. So we conclude that $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$ is a particular solution to the system $(*)$. The general solution is given by adding the particular solution to an arbitrary element of the nullspace:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for any numbers α and β.

PROBLEM 3

Consider the matrix $B = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 4 \\ 3 & 1 & 9 \end{bmatrix}$.

(1) Put the matrix in <u>row echelon form</u>, and compute a basis for its column space. *(15 pts)*

Solution: Subtract 2 times row 1 from row 2, and 3 times row 1 from row 3:

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 4 \\ 3 & 1 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -6 \\ 0 & -2 & -6 \end{bmatrix}$$

Then, let us subtract row 2 from row 3:

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -6 \\ 0 & -2 & -6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Now from this row echelon form we can see the first 2 columns are pivot columns, so we get that the first 2 columns of B give a basis of the column space, i.e.

$$\text{a basis of } C(B) \text{ cosnists of the vectors } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(2) Find a linear combination of the columns of B which is 0. *(10 pts)*

Solution: Any linear combination of the columns which is 0 will be accepted. But one way to work this out methodically is to recall that:

$$\alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} = 0$$

precisely means that:

$$B \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

so we are looking for a non-zero vector in the nullspace of B. This is the same as the nullspace of the row echelon form matrix, so we are looking for α, β, γ which satisfy:

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \qquad \text{i.e.} \qquad \begin{cases} \alpha + \beta + \gamma = 0 \\ -2\beta - 6\gamma = 0 \end{cases}$$

You can get a solution by setting γ equal to anything (say, equal to 1) and then using the equations above to solve for $\beta = -3$ and $\alpha = -2$.

(3) Compute the matrix $S = B^T B$ <u>and</u> the difference $S^T - S$. *(5 pts)*

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 5 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 4 \\ 3 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 4 & 40 \\ 4 & 2 & 14 \\ 40 & 14 & 122 \end{bmatrix}$$

This matrix is symmetric as $(B^T B)^T = B^T (B^T)^T = B^T B$, hence $S^T - S = 0$.

(4) Explain why for any 3×3 matrix X, the product XB cannot be invertible. *(5 pts)*

Solution: The rows of XB are linear combinations of the rows of B, which we have already seen has rank 2. So the 3×3 matrix XB does not have full column rank, hence it cannot be invertible.